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Discrete Applied Mathematics 81 (1998) 59–68

DISCRETE APPLIED MATHEMATICS

The maximal clique and colourability of curve contact graphs ☆

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Received 1 April 1996; received in revised form 9 April 1997; accepted 21 April 1997

Abstract

Contact graphs are a special kind of intersection graphs of geometrical objects in which the objects are not allowed to cross but only to touch each other. Contact graphs of simple curves, and line segments as a special case, in the plane are considered. The curve contact representations are studied with respect to the maximal clique and the chromatic number of the represented graphs. All possible curve contact representations of cliques are described, and a linear bound on chromatic number in the maximal clique size is proved for the curve contact graphs.

1. Intersection and contact graphs

The *intersection graph* of a set family $\mathcal{M}$ is defined as a simple graph G with the vertex set $V(G)=\mathcal{M}$ and the edge set $E(G)=\{\{A,B\}\subseteq\mathcal{M} \mid A\neq B,\ A\cap B\neq\emptyset\}$. Intersection graphs of geometrical objects attract much attention, owing to their various practical applications. For us, it is interesting to mention several works on the intersection graphs of curves or line segments in the plane [1, 9, 10].

A special type of geometrical intersection graphs – the *contact graphs*, in which the geometrical objects are not allowed to cross but only to touch each other, are considered. Unlike the general intersection graphs, only a few results are known in this field. There is a nice old result of Koebe [8], concerning representations of planar graphs as contact graphs of circles in the plane. In [3] a similar result about contact graphs of triangles is derived. The contact graphs of line segments are considered in works of de Fraysseix, de Mendez, Pach, and of Thomassen. It is proved that every bipartite planar graph is a contact graph of vertical and horizontal line segments [2],

☆ This work was supported by Czech research grants GAUK 193/1996 and 194/1996, GAČR 0194/1996, and CS-US Science and Technology Research grant 94051.

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PII S0166-218X(97)00075-9

and for general contact graphs of line segments, with contact of 2 segments in one contact point, a characterization is given in [13].

Following the ideas of intersection graphs of curves and of contact graphs of segments, we define contact graphs of simple curves in the plane, with contacts of more curves in one contact point allowed (only "one-sided" contacts), see also [6, 7]. Our paper deals with the curve contact representations of complete graphs, and the maximal cliques and chromatic number in the curve contact graphs. A related basic paper [7] concerns inclusions among various classes of the curve contact graphs and the recognition problem for them. The complete proofs may be also found in technical report [5].

2. Basic concepts of curve contact graphs

Simple curves of finite length (Jordan curves) in the plane are considered. Each curve has two *endpoints* and all of its other points are called interior points; they form the *interior* of the curve. We say that a curve φ *ends in* (*passes through*) a point X if X is an endpoint (interior point) of φ.

Definition. A finite set $\mathscr{R}$ of curves in the plane is called *curve contact representation* of a graph $\boldsymbol{G}$ if interiors of any two curves of $\mathscr{R}$ are disjoint and $\boldsymbol{G}$ is the intersection graph of $\mathscr{R}$. The graph $\boldsymbol{G}$ is called the *contact graph* of $\mathscr{R}$ and denoted by $\boldsymbol{G}(\mathscr{R})$. A curve contact representation $\mathscr{R}$ is said to be a *line segment contact representation* if each curve of $\mathscr{R}$ is a line segment. A graph $\boldsymbol{H}$ is called a *contact graph of curves* (*contact graph of line segments*) if there exists a curve contact representation (line segment contact representation) $\mathscr{S}$ such that $\boldsymbol{H} \cong \boldsymbol{G}(\mathscr{S})$.

A curve contact representation is called simply a *representation,* a contact graph of curves simply a *contact graph.* Any subset $\mathscr{S} \subseteq \mathscr{R}$ is called a *subrepresentation* of $\mathscr{R}$. A point C is a *contact point* of a representation $\mathscr{R}$ if it is contained in at least two curves of $\mathscr{R}$, and its *degree* is the number of curves of $\mathscr{R}$ containing C; a contact point of degree k is called a *k-contact point.*

In Fig. 1, an example of a curve contact representation and its contact graph are given. For a better view, every contact point is emphasized by a circle around it. Note that for any k-contact point C either all k curves containing C end in C or one curve is passing through C and the other $k-1$ curves end in C.

For our research, it is important to distinguish between "one-sided" and "two-sided" contact points – whether the other curves of the contact point are only on one side of the passing curve, or on both sides of it, see Fig. 2 (not every two-sided contact graph has a one-sided contact representation!). We may formally define a *one-sided contact point* as a contact point C in which either all of its curves end, or there exists a curve ϱ passing through C such that for all other curves $\sigma_1, \ldots, \sigma_{k-1}$ ending in C, the cyclic order of the curves outgoing from C is $\varrho, \varrho, \sigma_1, \ldots, \sigma_{k-1}$. In this paper we

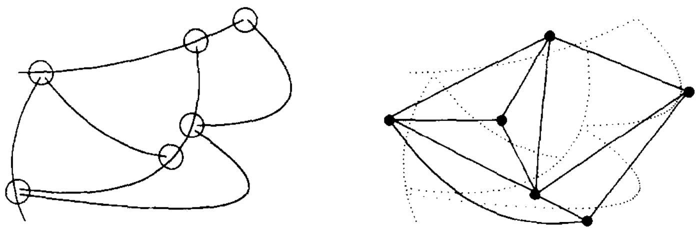


Fig. 1. An example of a curve contact representation of a graph.

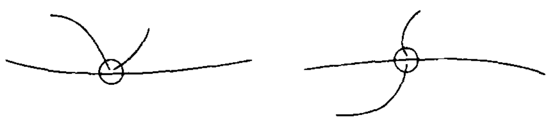


Fig. 2. The difference between one-sided and two-sided contact points.

will consider *only one-sided contact representations*, that seem to be better reflecting the natural meaning of a contact.

A representation $\mathscr{R}$ is called a *k-contact representation* if each contact point of $\mathscr{R}$ has degree at most k, the same definition is applied to line segment representations. A representation $\mathscr{R}$ is said to be *simple* if each pair of curves from $\mathscr{R}$ has at most one common contact point. Both these properties of representations are transferred to contact graphs, and we refer to them as k-contact or simple contact graphs in the obvious sense. It is clear that every line segment contact representation is simple; also every 2 or 3-contact representation can be rearranged to be simple but there exist 4-contact graphs with no simple contact representations, see [7].

We say that two contact representations $\mathscr{R}, \mathscr{S}$ are *similar* if there are bijections f between $\mathscr{R}$ and $\mathscr{S}$, and g between the contact points of $\mathscr{R}$ and $\mathscr{S}$, such that ϱ ends in (passes through) X iff $f(\varrho)$ ends in (passes through) $g(X)$. Similarity of representations clearly implies that the contact graphs are isomorphic. The next theorem enables us to handle a curve contact representation easier and to describe it using polynomial space (see [7] for the proof and for another possible description of a contact representation by the incidence graph).

Theorem 1. *For each contact representation $\mathscr{R}$, $|\mathscr{R}| = n$, there exists a representation $\mathscr{S}$ similar to $\mathscr{R}$, so that each curve from $\mathscr{S}$ is a piecewise linear curve with its vertices embedded on a grid of size* $\mathrm{O}(n) \times \mathrm{O}(n)$.

With this nice embedding of a contact representation at hand, it is not difficult to find a planar drawing for any 3-contact graph (details can be found in [5]):

Lemma 2.1. *Every* 3-*contact graph of curves is planar.*

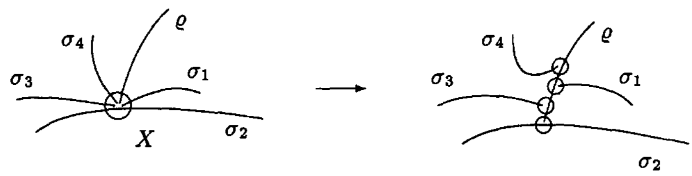


Fig. 3. An example of splitting X along the curve ϱ.

Some of the further proofs use the following two operations to change a contact representation (and consequently the contact graph): A trivial operation is *removing* a curve from a contact point. A more involved one is the operation of *splitting* a contact point X of curves $\varrho, \sigma_1, \dots, \sigma_k$ along the curve ϱ – it produces a contact representation in which the contact point X is replaced by k new 2-contact points $X_1, \dots, X_k$ of the pairs of curves $\varrho\sigma_1, \dots, \varrho\sigma_k$ (see Fig. 3). The representation obtained is, for simplicity, referred by the same symbols as the original one. Clearly, using the piecewise-linear embedding from Theorem 1, these operations may be applied to any contact representation[1].

3. Contact representations of complete graphs

It is easy to represent any complete $\boldsymbol{K}_m$ by a line segment contact representation, consisting of a "star" of m segments with a common endpoint. However, we study the problem of which complete graphs are representable with bounded contact degrees. The largest possible complete contact graphs are shown in Fig. 4, their maximality is proved next. In fact, it is derived that only two general types (with an exception of $\boldsymbol{K}_4$) of contact representations of cliques are possible, one of them is a simple representation.

Theorem 2. *The largest m, for which $\boldsymbol{K}_m$ is a k-contact (simple k-contact) graph of curves, is $m = \lfloor \frac{3}{2}k \rfloor$ ($m = k+1$) for $k \geqslant 3$; with an exception of $\boldsymbol{K}_4$ that is a simple 2-contact graph.*

The largest m, for which $\boldsymbol{K}_m$ is a k-contact graph of line segments, is $m = k+1$ for all $k \geqslant 2$.

The theorem is proved by the following sequence of observations and lemmas.

Observation. In Fig. 4, there are shown schemes (for every $k \geqslant 2$) of a line segment k-contact representation of $\boldsymbol{K}_{k+1}$, of a simple 2-contact representation of $\boldsymbol{K}_4$, and of a k-contact representation of $\boldsymbol{K}_{\lfloor \frac{3}{2}k \rfloor}$.

Observation. If there were a line segment 2-contact representation $\mathscr{R}$ of the graph $\boldsymbol{K}_4$, it would contain 6 contact points. But the convex hull of $\mathscr{R}$ must be polygon with at

[1] Note that we cannot apply them to general two-sided contact representations.

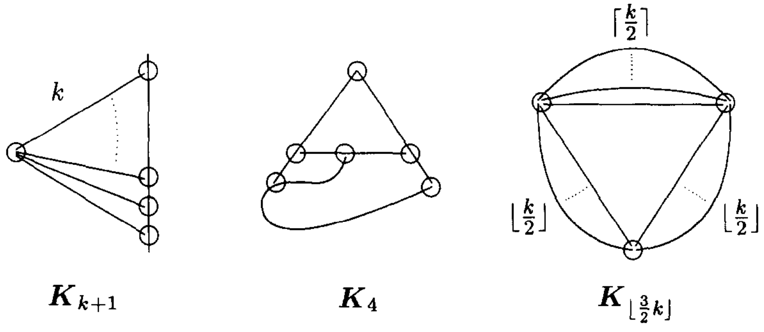


Fig. 4. Contact representations of complete graphs.

least 3 vertices, and each of these vertices either is a free endpoint of a segment, or is a contact point with 2 ending segments. Thus the representation $\mathscr{R}$ needs at least $6+3=9$ endpoints of only 4 segments, a contradiction.

Lemma 3.1. *For $k \geqslant 3$, the graph $\boldsymbol{K}_{k+2}$ is not a simple k-contact graph.*

Proof. Let us suppose that there exists a simple k-contact representation $\mathscr{R}$ of the graph $\boldsymbol{K}_{k+2}$. If there is no contact point of degree at least 3 in $\mathscr{R}$, we have a 2-contact subrepresentation of the non-planar graph $\boldsymbol{K}_5$. Otherwise we take 3 curves $\varrho_1, \varrho_2, \varrho_3$ with a common contact point X. Since the contact degree of X is at most k, there exist two other curves σ_1, σ_2 not containing X. By simplicity of $\mathscr{R}$, the curves $\varrho_1, \varrho_2, \varrho_3$ have pairwise no point in common except X, therefore $\varrho_1, \varrho_2, \varrho_3, \sigma_1, \sigma_2$ form a 3-contact subrepresentation of the graph $\boldsymbol{K}_5$, a contradiction to Lemma 2.1. □

The proof of the upper bound on the size of complete graphs representable by general k-contacts is more involved, and we divide it into a sequence of three lemmas. We say that a representation $\mathscr{R}$ contains a *three-bunch* of curves $\varrho_1, \varrho_2, \varrho_3$ with common contact points X, Y if $X \neq Y$ and $\{X, Y\} \subseteq \varrho_1 \cap \varrho_2 \cap \varrho_3$ (in fact, in such situation $\varrho_1 \cap \varrho_2 \cap \varrho_3 = \{X, Y\}$ holds, see the next lemma).

Lemma 3.2. *Let $\mathscr{R}$ be a contact representation containing a three-bunch $\varrho_1, \varrho_2, \varrho_3$ with contact points A, B, and let $\sigma \in \mathscr{R}$ be another curve touching all three curves $\varrho_1, \varrho_2, \varrho_3$. Then σ contains at least one of the points A, B.*

Proof (*sketch*). The validity of this technical lemma is almost evident. It can be proved by a contradiction – adding new curves α, β into contact points A, B, and then splitting A, B along α, β would produce a 3-contact representation of the non-planar graph $\boldsymbol{K}_{3,3}$. □

Lemma 3.3. *Let $\mathscr{R}$ be a contact representation, X, Y, Z some of its contact points, and $\varrho_1, \varrho_2, \ldots, \varrho_6 \in \mathscr{R}$ curves such that $X \in \varrho_1 \cap \varrho_2 \cap \varrho_3 \cap \varrho_4$ but $X \notin \varrho_5 \cup \varrho_6$, $Y \in \varrho_3 \cap \varrho_4 \cap$*

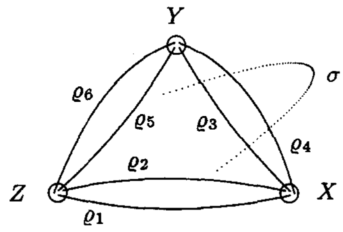


Fig. 5.

$\varrho_5 \cap \varrho_6$ but $Y \notin \varrho_1 \cup \varrho_2$, and $Z \in \varrho_5 \cap \varrho_6 \cap \varrho_1 \cap \varrho_2$ but $Z \notin \varrho_3 \cup \varrho_4$. If another curve $\sigma \in \mathscr{R}$ touches all the curves $\varrho_1, \ldots, \varrho_6$, then σ contains at least two of the points X, Y, Z.

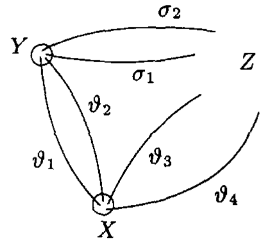
Proof. Note that $\{\sigma, \varrho_1, \varrho_2, \ldots, \varrho_6\}$ is a contact representation of the graph $\boldsymbol{K}_7$. The assumptions of the statement are schematically shown in Fig. 5. For a contradiction, suppose $X, Z \notin \sigma$.

We take the subrepresentation $\mathscr{S} = \{\sigma, \varrho_1, \varrho_2, \varrho_3, \varrho_5\} \subseteq \mathscr{R}$ of a graph $\boldsymbol{K}_5$ in which the contact points X, Y, Z have degrees at most 3. If there exists a contact point T of degree greater than 3 in $\mathscr{S}$, distinct from X, Y, Z, then in the case $T \in \sigma$ we split T along σ, otherwise we simply shorten some curve ending in T. Finally, we get a 3-contact representation $\mathscr{S}'$ of the graph $\boldsymbol{K}_5$, a contradiction to Lemma 2.1. □

Lemma 3.4. *For any k-contact representation $\mathscr{R}$ of the graph $\boldsymbol{K}_m$, $m \geqslant k+2$, $k \geqslant 4$, there exist contact points $X, Y, Z \in T(\mathscr{R})$ such that each curve of $\mathscr{R}$ contains at least two of the points X, Y, Z.*

Proof. First we consider the case when the representation $\mathscr{R}$ contains no three-bunch of curves. Since the graph $\boldsymbol{K}_6$ is not a 3-contact graph, we may suppose that there exists a k-contact point X, $k \geqslant 4$, in $\mathscr{R}$; and $\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4 \in \mathscr{R}$ are some curves containing $X, \sigma_1, \sigma_2 \in \mathscr{R}$ some curves not containing X. The subrepresentation $\mathscr{S}_1 = \{\sigma_1, \sigma_2, \vartheta_1, \vartheta_2, \vartheta_3\}$ of the graph $\boldsymbol{K}_5$ is not 3-contact. If $\mathscr{S}_1$ contains a common contact point of the curves $\vartheta_1, \vartheta_2, \vartheta_3$, these curves form a three-bunch in $\mathscr{R}$ and that case will be discussed later. Otherwise, $\mathscr{S}_1$ contains a 4-contact point Y of curves, say, $\vartheta_1, \vartheta_2, \sigma_1, \sigma_2$ (Fig. 6). Then the subrepresentations $\mathscr{S}_2 = \{\vartheta_1, \vartheta_3, \vartheta_4, \sigma_1, \sigma_2\}$ must also contain a 4-contact point Z. If $Z \in \vartheta_1$ (ϑ_2), we again get a three-bunch of curves either $\vartheta_1, \vartheta_3, \vartheta_4$ or $\vartheta_1, \sigma_1, \sigma_2$. In the last case $Z \in \vartheta_3 \cap \vartheta_4 \cap \sigma_1 \cap \sigma_2$ we may directly apply Lemma 3.3, and the lemma is proved.

Thus it remains to finish the case when the representation $\mathscr{R}$ contains a three-bunch of curves $\varrho_1, \varrho_2, \varrho_3$ with common contact points X, Y, see Fig. 6 on the right. We denote by $\mathscr{M}$ the set of all curves of $\mathscr{R}$ containing both points X, Y, by $\mathscr{N}_X$ the set of all curves containing X but not Y, and by $\mathscr{N}_Y$ the set of all curves containing Y but not X. By Lemma 3.2, $\mathscr{M} \cup \mathscr{N}_X \cup \mathscr{N}_Y = \mathscr{R}$. Since $m = |\mathscr{R}| \geqslant k+2$, $|\mathscr{N}_X| \geqslant 2$



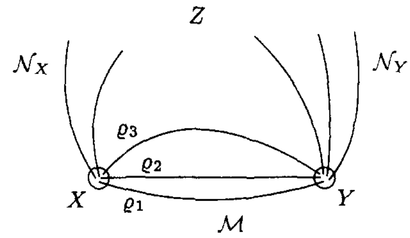


Fig. 6.

and $|\mathcal{N}_Y| \geqslant 2$ and we choose some $\omega_1, \omega_2 \in \mathcal{N}_X$, $\pi_1, \pi_2 \in \mathcal{N}_Y$. The representation $\mathcal{R}'$, obtained from $\mathcal{R}$ by removing the curves ϱ_1, ϱ_2 from all contact points except X, Y, still represents the graph $\boldsymbol{K}_m$. If the curves $\omega_1, \omega_2, \pi_1, \pi_2$ have no common contact point, then $\{\varrho_1, \omega_1, \omega_2, \pi_1, \pi_2\} \subseteq \mathcal{R}'$ forms a 3-contact subrepresentation of $\boldsymbol{K}_5$, a contradiction. Otherwise, we denote some $Z \neq X, Y$, $Z \in \omega_1 \cap \omega_2 \cap \pi_1 \cap \pi_2$, and now Lemma 3.3 may be applied. This proves the statement for $\mathcal{R}'$, hence also for the original representation $\mathcal{R}$. □

4. Clique, independence and chromatic number of contact graphs

Many graph problems that are hard in the general case, can be solved quickly for special intersection graphs. For example, it is easy to find the chromatic number, the maximal clique or the maximal independent set of an interval graph. We show that the maximal clique of a curve contact graph can be found in polynomial time if its contact representation is given, while the chromatic number and the independent set size remain NP-complete under the same assumption.

Based on the results of the previous section, it is easy to describe all possible shapes of cliques in a contact representation $\mathcal{R}$ of a graph.

Theorem 3. *If $\mathcal{S} \subseteq \mathcal{R}$, $|\mathcal{S}| \neq 4$, is a subrepresentation of a clique in the contact graph $\boldsymbol{G}(\mathcal{R})$, then either there is a contact point contained in all curves of $\mathcal{S}$ except at most one, or there are three contact points such that each curve of $\mathcal{S}$ contains at least two of them.*

Proof. It is straightforward from Lemmas 3.1, 3.4. □

Corollary 4.1. *There exists a polynomial algorithm that for given contact representation $\mathcal{R}$ of a graph $\boldsymbol{G}$ finds the maximal clique of $\boldsymbol{G}$.*

Proof. The algorithm is an easy consequence of Theorem 3. It examines sequentially all the contact points for an existence of the first type clique, and then all triples of contact points for an existence of the second type clique. If the clique found is smaller than 4, the algorithm must also check all quadruples of vertices for the 4-clique.

The only problem is in the form of the input representation (not to be too large) – either it can be given as the embedding from Theorem 1, or better in the form of the incidence graph (describing the incidences between curves and contact points, see [7]). The running time of this simple algorithm is then $O(n^4)$ where $n = |V(G)|$. □

Proposition 4.2. *The INDEPENDENT SET and the* 3-*COLOURABILITY problems are* NP-*complete for contact graphs* (2-*contact graphs*) *even when the contact representation is given.*

Proof (*sketch*). It is well known that the INDEPENDENT SET and the 3-COLOURABILITY problems remain NP-complete even for planar graphs with vertices of degrees at most 4, see [4]. By [7], such graphs are always 2-contact graphs, and their contact representation can be quickly constructed. □

Further we show that the contact graphs are "almost perfect", i.e. their chromatic number is bounded by a linear function of the maximal clique size. However, an infinite sequence of contact graphs, for which the chromatic number grows faster than the maximal clique, is constructed.

These results may be compared with other kinds of intersection graphs. The interval graphs, as a special case of perfect graphs, have always the chromatic number equal to the maximal clique. On the other hand, for intersection graphs of curves no bound on the chromatic number with respect to the maximal clique is known.

Lemma 4.3. *The chromatic number of a k-contact graph is at most* $2k$.

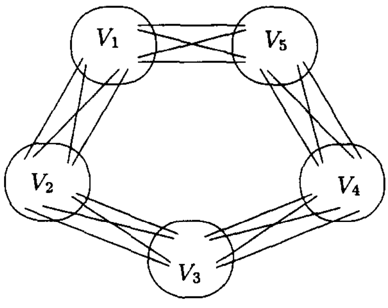
Proof. For $k = 2, 3$, the statement follows from Lemma 2.1 and the "Four-colour" theorem.

Otherwise we take an arbitrary k-contact representation $\mathscr{R}$ of a graph $\boldsymbol{G}$, $n = |V(\boldsymbol{G})|$, and for each curve $\varrho \in \mathscr{R}$ we denote by $i(\varrho)$ the number of other curves of $\mathscr{R}$ that end in interior points of ϱ. Obviously, $\sum_{\sigma\in\mathscr{R}} i(\sigma) \leqslant 2n$, because in each contact point only one curve can pass through, so each end of a curve is counted at most once. If there exists a curve $\sigma \in \mathscr{R}$ for which $i(\sigma) \leqslant 1$, the vertex σ in the graph $\boldsymbol{G}(\mathscr{R})$ has degree at most $2(k-1)+1 = 2k-1$, thus we may proceed by induction with a smaller representation $\mathscr{R}\setminus\{\sigma\}$ and then colour the vertex σ.

The remaining case is that for all $\sigma \in \mathscr{R}$, $i(\sigma) \geqslant 2$. From the inequality $\sum_{\sigma\in\mathscr{R}} i(\sigma) \leqslant 2n$ it is straightforward $i(\sigma) = 2$ for all σ, and in the sum an equality holds which means that each curve $\varrho \in \mathscr{R}$ must end in an interior point of another curve. Therefore the degrees of contact points are bounded by 3 and this is the case discussed first. □

Theorem 4. *For any contact graph* $\boldsymbol{G}$, $\chi(\boldsymbol{G}) \leqslant 2\cdot\omega(\boldsymbol{G})$; *while for every integer m there exists a contact graph* $\boldsymbol{H}_m$ *with* $\omega(\boldsymbol{H}_m) = m$ *and* $\chi(\boldsymbol{H}_m) \geqslant m + \lceil \frac{m-1}{4} \rceil$.

Proof. The first statement is a clear consequence of Lemma 4.3.



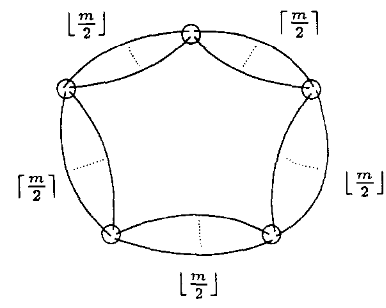


Fig. 7.

The graph $\boldsymbol{H}_m$ from the second statement is constructed on the vertex set $V_1 \cup V_2 \cup V_3 \cup V_4 \cup V_5$, where $|V_1| = |V_3| = |V_4| = \lfloor \frac{m}{2} \rfloor$ and $|V_2| = |V_5| = \lceil \frac{m}{2} \rceil$, as a union of complete graphs on the vertex sets $V_1 \cup V_2$, $V_2 \cup V_3$, $V_3 \cup V_4$, $V_4 \cup V_5$ and $V_5 \cup V_1$. An m-contact representation of this graph is presented in Fig. 7.

It is easy to check that $\omega(\boldsymbol{H}_m) = m$. We set $k = \lfloor \frac{m}{2} \rfloor$, $l = \lceil \frac{m}{2} \rceil$. Suppose $\boldsymbol{H}_m$ is coloured by $m+c$ colours. We denote by C_i the set of all colours occurring in vertices of V_i, $i = 1,2,3,4,5$ (remember that colours on V_i must be distinct, hence $|C_1| = |C_3| = |C_4| = k$, $|C_2| = |C_5| = l$), and set $C = C_1 \cup C_2 \cup C_3 \cup C_4 \cup C_5$, $D_1 = C \setminus C_1$, $D_5 = C \setminus C_5$. Then we know $C_2 \subset D_1$, $C_3 \cap C_2 = \emptyset$, so $|C_3 \cap D_1| \leqslant |D_1| - |C_2| = m+c-k-l = c$. Similarly, $|C_3 \cap D_5| \leqslant |D_5| - |C_4| = m+c-l-k = c$, and because $D_1 \cup D_5 = C$, we finally get $k = |C_3| \leqslant |C_3 \cap D_1| + |C_3 \cap D_5| \leqslant 2c$. Therefore $c \geqslant \lceil \frac{k}{2} \rceil \geqslant \lceil \frac{m-1}{4} \rceil$. □

5. Concluding remarks

We have shown a simple characterization of subrepresentations of cliques in contact graphs of curves in the plane. It is interesting to ask how this result depends on the topological structure of the plane – since our proofs were based on the fact that $\boldsymbol{K}_5$ and $\boldsymbol{K}_{3,3}$ are not planar graphs. However, considering surfaces of higher genus seems to change the situation only for low contact degrees:

For example, if we take simple contact representations of curves on the torus, we find a 2-contact representation of $\boldsymbol{K}_5$ and a 3-contact representation of $\boldsymbol{K}_7$, but to represent $\boldsymbol{K}_k$ for $k \geqslant 8$ we already need a $(k-1)$-contact simple representation. This observation can be proved by counting endpoints of the curves, without taking the topological structure into account. We think that for general contact graphs the situation is similar (compare with Lemma 3.4), so we conjecture a negative answer to the following question, concerning general contact graphs.

Problem. Is, for every k, the graph $\boldsymbol{K}_n$, $n = \lfloor \frac{3}{2}k \rfloor + 1$, a k-contact graph of curves on the torus (or on some surface of higher genus)?

We have also considered a relation between the maximal clique size and the chromatic number of a contact graphs. The proof of Lemma 4.3 uses the structure of the plane only for $k = 2, 3$ (this is necessary due to the existence of the above mentioned representations of K_5 and K_7 on the torus), for higher k it is independent of the topology. As proposed by A. Kostochka, it is possible to improve the bound of Lemma 4.3 to $\chi \leqslant 2k-1$ for k large enough. Generally, it seems that this bound can be improved much more, especially for simple contact representations.

Problem. Is it right that for a simple k-contact graph G, $\chi(G) \leqslant k+o(k)$ (or even $\chi(G) \leqslant k+\text{const}$)?

Problem. Determine $R = \limsup_{n\to\infty} \frac{c_n}{n}$, where c_k is the maximal chromatic number over all k-contact graphs (we know $1.25 \leqslant R \leqslant 2$).

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